CHAPTER ONE EXERCISES

EXERCISES 1-A

1. Express the following in terms of i:
(i) $\sqrt{-4}$ (ii) $\sqrt{-25}$ (iii) $\sqrt{-5}$ (iv) $\sqrt{-27}$ (v) $\sqrt{16}-\sqrt{-9}$

2. Express each of the following as an element of the set $\{-1, 1, i, -i\}$:
(i) i^3 (ii) i^6 (iii) i^9 (iv) i^{-4} (v) i^{16}.

3. Express each of the following in the form $a + ib$:
(i) $(3+2i)+(5-i)$ (ii) $(6-i)+(4-3i)$ (iii) $(-2+3i)+(6-4i)$
(iv) $(-2-i)+(-1+7i)$ (v) $6i+(3+5i)$ (vi) $(a+ib)+(c+id)$

4. Simplify each of the following:
(i) $(2-6i)-(1+i)$ (ii) $(3-6i)-(2+4i)$ (iii) $(2-i)-(-1+4i)$
(iv) $3-(2+4i)$ (v) $(6-2i)-4$ (vi) $(a+ib)-(2-3i)$

5. Express in the form $a + ib$:
(i) $(3+i)(2+4i)$ (ii) $(1-i)(2+3i)$ (iii) $(2-i)(3+2i)$
(iv) $(2+3i)(2-3i)$ (v) $(1-4i)^2$ (vi) $(2+i)^3$

6. If $z_1 = 3 - i$, $z_2 = 1 + 2i$ and $z_3 = -2i$, express in the form $a + ib$:
(i) $3z_1$ (ii) $z_1 - z_3$ (iii) $2z_1 + z_2$ (iv) $2z_2 + z_3$
(v) $-2z_2$ (vi) iz_2 (vii) $2z_1 + iz_3$ (viii) $i(z_2.z_3)$

7. If $z = -3 + 5i$, find (i) $z + \bar{z}$ (ii) $z.\bar{z}$

8. Express the following in the form $x + iy$:
(i) $\dfrac{1}{2-3i}$ (ii) $\dfrac{2+i}{1-2i}$ (iii) $\dfrac{3+2i}{2-3i}$
(iv) $\dfrac{2i}{2+i}$ (v) $\dfrac{3-2i}{i}$ (vi) $\dfrac{2-3i}{2+i}$

9. If $z = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$, find z^2.

Hence verify that $\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^3 = 1$.

10. If $z_1 = 2 - 5i$ and $z_2 = 1 - 4i$, express in the form $a + ib$:
(i) $z_1 + \bar{z}_2$ (ii) $z_1\bar{z}_2$ (iii) $\dfrac{1}{\bar{z}_1}$ (iv) $i\bar{z}_1 z_2$

11. If $z = 3 - i$, express $z + \dfrac{1}{z}$ in the form $a + ib$.

12. If $z = x + iy$, prove that

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z}) \text{ and } \text{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

13. (i) Evaluate $1 + i - 3i^2 + i^7$

(ii) Given that $(2+3i)z = 4 - i$, find the complex number z in the form $a + ib$.

14. Show that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$.

15. Express (i) $\dfrac{1+i}{1-i}$ and (ii) $\dfrac{1-i}{1+i}$ in the form $a+ib$.

Hence, or otherwise, find k if $\dfrac{1+i}{1-i} = k\left(\dfrac{1-i}{1+i}\right)$.

16. Express $\dfrac{-1+i\sqrt{3}}{-1-i\sqrt{3}}$ in the form $a+ib$, where a and b are real numbers.

17. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, show that $z_1\bar{z}_2 + \bar{z}_1 z_2$ is real.

18. (i) Simplify $(1-i)^{-2} + (1+i)^{-2}$.

(ii) The complex numbers u, v and w are such that $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{w}$.

If $u = 3+2i$ and $v = 2+3i$, find w in the form $a+ib$.

19. Simplify the following

(a) $(2+6i)+(9-2i)$

(b) $(8-3i)-(1+5i)$

(c) $3(7-3i)+i(2+2i)$

(d) $(3+5i)(1-4i)$

(e) $(5+12i)(6+7i)$

(f) $(2+i)^2$

(g) i^3

(h) i^4

(i) $(1-i)^3$

(j) $(1+i)^2+(1-i)^2$

(k) $(2+i)^4+(2-i)^4$

(l) $(a+ib)(a-ib)$

20. The imaginary part of a complex number is known to be twice its real part. The absolute value of this number is 4. Which number is this?

21. The real part of a complex number is known to be half the modulus of that number. The imaginary part of the number is 1. Which number is it?

22. True or False? (In mathematics this means that you should either give a proof that the statement is always true, or else give a counterexample, thereby showing that the statement is not always true.)

For any complex numbers z and w one has

(a) $\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(z+w)$

(b) $\overline{z+w} = \bar{z} + \bar{w}$

(c) $\mathfrak{Im}(z) + \mathfrak{Im}(w) = \mathfrak{Im}(z+w)$

(d) $\overline{zw} = (\bar{z})(\bar{w})$

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w) = \mathfrak{Re}(zw)$

(f) $\overline{z/w} = (\bar{z})/(\bar{w})$

(g) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$

(h) $\mathfrak{Re}(iz) = i\mathfrak{Re}(z)$

(i) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$

(j) $\mathfrak{Re}(iz) = i\mathfrak{Im}(z)$

(k) $\mathfrak{Im}(iz) = \mathfrak{Re}(z)$

(l) $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$

23. Show that $\left|e^{a+bi}\right| = e^{a}$.

EXERCISES 1-B

1. Find, in the form $a + ib$, the roots of these equations:
(i) $z^2 + 6z + 10 = 0$ (ii) $z^2 - 2z + 2 = 0$
(iii) $z^2 - 6z + 13 = 0$ (iv) $2z^2 - 2z + 5 = 0$

2. Verify that $5 + i$ is a root of the equation $z^2 - 10z + 26 = 0$ and write down the other root.

3. Write down the quadratic equation with roots
(i) $\pm 2i$ (ii) $1 \pm 2i$ (iii) $3 \pm 2i$ (iv) $-2 \pm i\sqrt{5}$.

4. Verify that $\frac{1}{2} + \frac{1}{2}i$ is a root of the equation $2z^2 - 2z + 1 = 0$ and write down the other root.

5. Solve for x and y in each of the following equations:
(i) $x + iy = (2 - 3i)(3 + i)$ (ii) $2x + 5iy = (6 + 2i)(3 - 4i)$
(iii) $(x + iy) + 3(2 - 3i) = 6 - 10i$ (iv) $2x + iy = 6$
(v) $(2x + y - 5) + i(3x + y - 7) = 0$ (vi) $x + iy = (3 - 2i)(3 + 2i)$

6. If $\dfrac{3 - 2i}{5 + i} = a + ib$, find the value of a and b.

7. Find the values of x and y in these equations:
(i) $(3 - 5i) - (x + iy) = (6 + i) + (y - xi)$
(ii) $(x + iy)(2 + i) = (1 - i)^2$

8. If $x^2 + 2xyi + y^2 = 10 + 6i$, find the values of x and y.

9. If $(x + iy)^2 = 8 - 6i$, find the values of x and y.

10. Express each of the following in the form $a + ib$:
(i) $\sqrt{5 + 12i}$ (ii) $\sqrt{-15 + 8i}$ (iii) $\sqrt{9 - 40i}$

11. Express $\sqrt{2 - 2i\sqrt{3}}$ in the form $a + ib$, where $a, b \in R$.

12. Expand $(x + iy)(x - iy)$ and hence factorise
(i) $x^2 + 4$ (ii) $x^2 + 9y^2$ (iii) $4x^2 + 25y^2$

13. Given that $\dfrac{5}{x + iy} + \dfrac{2}{1 + 3i} = 1$, find x and y, both $\in$ R.

14. Given that $z = 1 + i$, show that $z^3 = -2 + 2i$.
For this value of z, the real numbers p and q are such that

$$\frac{p}{1+z} + \frac{q}{1+z^3} = 2i$$

Find the values of p and q.

15. (i) One root of the equation $x^2 - ax - b = 0$ is $2 - i$.
Find the values of a and $b \in R$.

(ii) Two complex numbers z_1 and z_2 are such that $z_1 + z_2 = 1$.

If $z_1 = \dfrac{x}{1+i}$ and $z_2 = \dfrac{y}{1+2i}$, find x and y.

16. The points a and c represent in the Argand diagram the roots of the equation $z^2 - 6z + 13 = 0$. If $[ac]$ is the diagonal of a square $abcd$, find
(i) the numbers represented by the points b and d, if d has the larger x-coordinate.
(ii) the area of the square $abcd$.

17. If $z_1 = a + b^2 - 3i$ and $z_2 = 2 - ab^2 i$, find the real values of a and b such that $z_1 = \bar{z}_2$.

EXERCISES 1-C

1. Represent each of the following numbers on an Argand diagram:
(i) $2 + 3i$ (ii) $-2 + i$ (iii) $-3i$ (iv) 4
(v) $i(2 - 3i)$ (vi) $\dfrac{3}{i}$ (vii) $(2 + 3i)(i - i)$ (viii) $\dfrac{3 - 2i}{3 + 4i}$

2. Write down the modulus of each of these:
(i) $3 - 4i$ (ii) $1 + 2i$ (iii) $\sqrt{3} - i$
(iv) $(2 + 3i)(1 - 2i)$ (v) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$ (vi) $(x + 1) + iy$

3. Express each of the following in the form $a + ib$ and hence find the modulus of each:
(i) $i(1 - i)$ (ii) $(1 + i)(\sqrt{3} + i)$ (iii) $\dfrac{-4}{1+i}$

4. If $z_1 = 3 - 4i$ and $z_2 = 5 + 12i$, show that $|z_1 + z_2| < |z_1| + |z_2|$.

5. If $z_1 = 3 - i$ and $z_2 = 4 - 3i$ verify that

(i) $|z_1|.|z_2| = |z_1 z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

6. Express each of these complex numbers in the form $r(\cos\theta + i\sin\theta)$:
(i) $1 + i$ (ii) $\sqrt{3} + i$ (iii) $-\sqrt{2} + i\sqrt{2}$

(iv) $1 - i\sqrt{3}$ (v) $4i$ (vi) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(vii) -5 (viii) $-3i$ (ix) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

7. Simplify the following and express your answers in the form $r(\cos\theta + i\sin\theta)$:

(i) $(1 + i\sqrt{3})^2$ (ii) $\frac{-2}{-\sqrt{3} + i}$

8. If $z = x + iy$, evaluate $\left|\frac{z-1}{1-z}\right|$

$\left(\text{Hint: } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.\right)$

9. Express each of the following in the form $a + ib$:

(i) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ (ii) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(iii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (iv) $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

10. Find the modulus and argument of z, where $z = \frac{1}{(1-i)^2}$.

11. Express the following in the general modulus-argument form:

(i) $2i$ (ii) $1 + i\sqrt{3}$ (iii) $-3 - i\sqrt{3}$ (iv) $\frac{2}{-1+i}$

12. Find the modulus and argument of each root of the equation $x^2 + 4x + 8 = 0$.

13. Find the modulus and argument of (i) wz and (ii) $\frac{w}{z}$ given that $w = 10i$ and $z = 1 + i\sqrt{3}$.

14. Represent the following graphically and write it in polar form.

a) $1 + j\sqrt{3}$ b) $\sqrt{2} - j\sqrt{2}$

15. Represent the following graphically and write it in rectangular form.

a) $6(\cos 180° + j\sin 180°)$ b) $7.32 \angle -270°$

16. compute and draw the following numbers in the complex plane

a) $e^{\pi i/3}$ b) $\sqrt{2}e^{3\pi i/4}$ c) $e^{\pi i} + 1$ d) $e^{i\ln 2}$ e) $\frac{e^{-\pi i}}{e^{\pi i/4}}$ f) $\frac{1}{e^{\pi i/4}}$ g) $12e^{\pi i} + 3e^{-\pi i}$

17. compute the absolute value and argument of $e^{(\ln 2)(1+i)}$

18. Write the following numbers in standard (rectangular) form.

(a) $3e^{\frac{3\pi}{4}i}$ (b) $12e^{-\frac{22\pi}{3}i}$ (c) $19e^{\frac{14\pi}{2}i}$

19. Given $Z = 5 + 2i$, $w = -3+5i$ and $V = 7i$

(a) Plot the complex numbers $z + \bar{z}$, $w + \bar{w}$ and $v + \bar{v}$ in the complex plane.

(b) Plot the complex number $2w + \bar{z} + v$ in the complex plane.

(c) Plot the complex number $v - z - w$ in the complex plane.

20. Find $[r, \theta]$ in the following and express your answer in the polar form.

(a). $[2, 1/4\,\pi] \times [r, \theta] = [10, 5/4\,\pi]$ (b). $[r_1, \theta_1] \times [r, \theta] = [r_2, \theta_2]$ (c). $[r_2, \theta_2] / [r_1, \theta_1] = [r, \theta]$

EXERCISES 1-D

1. Use the result $z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$ to simplify the following, giving your answers in the form $a + ib$:

(i) $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (ii) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2$

(iii) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (iv) $\left[2\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)\right]^3$

2. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

3. Use the result obtained in question 2 to express each of the following in the form $a + ib$:

(i) $\dfrac{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$ (ii) $\dfrac{\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

(iii) $\dfrac{\left[2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]^4}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$ (iv) $\dfrac{1}{\left[\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^4}$

4. Simplify $\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^2$.

5. By expressing each of the following in the form $r(\cos\theta + i\sin\theta)$, write down the modulus and argument of each of these:

(i) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$ (ii) $3\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

(iii) $\sqrt{2}(\cos\theta - i\sin\theta)$ (iv) $\left[2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right]^2$

6. (i) Show that if $z = \cos\theta + i\sin\theta$, then $\bar{z} = \dfrac{1}{z}$.

(ii) If $z = \cos\theta - i\sin\theta$, write down the real part and the imaginary part of $(1 - z)$.

Now show that $\text{Re}\left(\dfrac{1-z}{1+z}\right) = 0$, where Re is the real part of a complex number.

7. Find, in the form $a+ib$, the complex number z such that

$$z\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=1.$$

8. Express $\dfrac{1+i\tan\theta}{1-i\tan\theta}$ in the form $\cos k\theta+i\sin k\theta,\ k\in R$.

9. Express $\dfrac{-1+i\sqrt{3}}{2}$ in the form $r(\cos\theta+i\sin\theta)$

and hence simplify $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$.

10. Express the complex number z, given by $z=-2+2\sqrt{3}i$, in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-\pi<\theta\leqslant\pi$.
Hence, or otherwise, express z^3 in the form $a+ib$.

Find the modulus and argument of $\dfrac{1}{z^4}$.

11. If $z=\cos\theta+i\sin\theta$, show that $z+\dfrac{1}{z}=2\cos\theta$.
Hence find the value of $z-z^{-1}$.

12. Find the value of k if

$$(\cos\theta+i\sin\theta)^3+\frac{1}{(\cos\theta+i\sin\theta)^3}=k\cos 3\theta.$$

13. Express the following in the form *x* + i*y*

a $(\cos 2\theta+\mathrm{i}\sin 2\theta)(\cos 3\theta+\mathrm{i}\sin 3\theta)$

b $\left(\cos\frac{3\pi}{11}+\mathrm{i}\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11}+\mathrm{i}\sin\frac{8\pi}{11}\right)$

c $3\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\times 2\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos\left(\frac{-\pi}{12}\right)+\mathrm{i}\sin\left(\frac{-\pi}{12}\right)\right)\times\sqrt{3}\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

e $4\left(\cos\left(\frac{-5\pi}{9}\right)+\mathrm{i}\sin\left(\frac{-5\pi}{9}\right)\right)\times\frac{1}{2}\left(\cos\left(\frac{-5\pi}{18}\right)+\mathrm{i}\sin\left(\frac{-5\pi}{18}\right)\right)$

f $6\left(\cos\frac{\pi}{10}+\mathrm{i}\sin\frac{\pi}{10}\right)\times 5\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\times\frac{1}{3}\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)$

g $(\cos 4\theta+\mathrm{i}\sin 4\theta)(\cos\theta-\mathrm{i}\sin\theta)$

h $3\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)\times\sqrt{2}\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)$

(i) $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)}$ **(j)** $\dfrac{3\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)}$ **(k)** $\dfrac{\cos 2\theta-\mathrm{i}\sin 2\theta}{\cos 3\theta+\mathrm{i}\sin 3\theta}$ **(l)** Evaluate $\dfrac{\left(\cos\frac{7\pi}{13}+\mathrm{i}\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13}-\mathrm{i}\sin\frac{4\pi}{13}\right)^6}$

14. Use de Moivre's theorem to simplify each of the following:

a $(\cos\theta + i\sin\theta)^6$

b $(\cos 3\theta + i\sin 3\theta)^4$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

f $\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)^{15}$

g $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$

h $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

i $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

j $\dfrac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

k $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta + i\sin 3\theta}$

l $\dfrac{\cos\theta - i\sin\theta}{\cos 2\theta - i\sin 2\theta}$

m $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta}$

n $\dfrac{\cos\theta - i\sin\theta}{\cos 4\theta - i\sin 4\theta}$

o $\dfrac{\cos 2\alpha + i\sin 2\alpha}{\cos\alpha + i\sin\alpha}$.

p $\dfrac{(\cos\theta - i\sin\theta)^2}{(\cos\theta + i\sin\theta)^3}$

(q) $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$

(r) $(\cos 2\theta + i\sin 2\theta)^3\,(\cos 3\theta + i\sin 3\theta)^2$

EXERCISES 1-E

1. Show that $-2+4i$ is a root of the equation $z^2+4z+20=0$ and write down the other root.

2. Solve these equations, giving the roots in the form $a+ib$.

(i) $z^2-2z+17=0$ (ii) $z^2+4z+7=0$.

3. If $z = x+iy$, verify that

(i) $z+\bar{z} = 2\,\mathrm{Re}(z)$ (ii) $|z|^2 = z.\bar{z}$

4. If $z = x+iy$, verify that

(i) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ii) $z_1\bar{z}_2 - \bar{z}_1 z_2 = 2i\,\mathrm{Im}(z_1\bar{z}_2)$

(iii) $\mathrm{Re}(z_1\bar{z}_2) = \frac{1}{2}(z_1\bar{z}_2 + \bar{z}_1 z_2)$

5. Show that $1+i$ is a root of the equation $z^3-4z^2+6z-4=0$ and find the other two roots.

6. Form the quadratic equation whose roots are $-2\pm i$.
Now show that $-2+i$ is a root of the equation $z^3+z^2-7z-15=0$ and find the other roots.

7. Form the quadratic equation whose roots are $-3 \pm 2i$.
Hence form the cubic equation whose roots are $-3 \pm 2i$ and 1.

8. (i) Form the quadratic equation, one of whose roots is $3 - i$.

(ii) Form the cubic equation, two of whose roots are 2 and $-1 + i$.

9. Find the real root of the equation $z^3 + z + 10 = 0$, given that one root is $1 - 2i$.

10. $\dfrac{1+2i}{1-i}$ is a root of the quadratic equation $ax^2 + bx + 5 = 0$, where $a,b \in R$.

Find the values of a and b.

11. Verify that $2i$ is a root of the equation $x^4 + 2x^3 + 7x^2 + 8x + 12 = 0$ and find the other roots.

12. Factorise $z^3 - 1$ and hence solve the equation $z^3 - 1 = 0$, giving the complex roots in the form $a \pm ib$.

If the complex roots are α and β, verify that $\alpha^2 = \beta$.

13. If $2 + i$ is a root of the equation $z^3 + az^2 + bz + 10 = 0$, find the values of a and b given that the product of the three roots is -10.

14. Explain why the roots of the equation

$$z^2 - (3+2i)z + 1 + 3i = 0$$

do not occur in conjugate pairs.

Now use the formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to express the roots of the given equation in the form $a + ib$.

15. Given that $\sqrt{24 - 10i}$ is $\pm(5 - i)$, solve the equation $z^2 + (1 - 3i)z - (8 - i) = 0$, expressing the roots in the form $a + ib$.

16. Obtain a quadratic function $f(z) = z^2 + az + b$, where a and b are real constants such that $f(-1-2i) = 0$.

17. Given that $\alpha = 1 + 3i$ is a root of the equation $z^2 - (p + 2i)z + q(1 + i) = 0$, and that p and q are real, find p and q and the other root of the equation.

18. Show that i is a root of the equation

$$z^3 - iz^2 - z + i = 0.$$

and, by factorising or otherwise, find the other two roots.

19. The complex number z satisfies the equation

$$4z\bar{z} - 8z = 13 - 4i$$

where $\bar{z}$ is the complex conjugate of z.
Find in the form $x + iy$, the two possible values of z.

20. Solve the equation $z^2 - (4+5i)z - 3 + 9i = 0$.

21. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. Given that the sum of the squares of the roots is 8, find the two pairs of values of p and q.

EXERCISES 1-F

1. Use De Moivre's theorem to simplify the following, expressing each answer in the form $a + ib$:

(i) $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^4$ (ii) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7$

(iii) $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^5$ (iv) $\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^6$

2. Express $\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$.
Hence use De Moivre's theorem to express $(\sqrt{3} + i)^8$ in the form $a + ib$.

3. Change to modulus-argument form and use De Moivre's theorem to evaluate each of the following, giving your answers in the form $a + ib$:

(i) $(1-i)^4$ (ii) $(-1-i)^7$ (iii) $(-2-2i)^5$ (iv) $(-\sqrt{3}-i)^3$

4. Use De Moivre's theorem to show that

(i) $(1+i)^8 = 16$ (ii) $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^9 = -i$

5. Express $\left(\frac{1 - i\sqrt{3}}{4}\right)^{12}$ in the form $a + ib$.

6. Express $\frac{-1 + i\sqrt{3}}{\sqrt{3} + i}$ in the form $a + ib$. Hence evaluate $\left(\frac{-1 + i\sqrt{3}}{\sqrt{3} + i}\right)^{99}$.

7. Use the identity $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$ to show that

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ (ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

8. Express $\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$ in the form $\cos\theta + i\sin\theta$, $\theta \in Z$, and hence express

$\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^8$ in the form $a + ib$.

9. Prove that $\left\{\cos\left(\theta + \frac{\pi}{3}\right) + i\sin\left(\theta + \frac{\pi}{3}\right)\right\}^6 = \cos 6\theta + i\sin 6\theta$.

10. Use De Moivre's theorem to evaluate each of the following:

(i) $\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^6$ (ii) $\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^4$

[Hint: $\sin\theta + i\cos\theta = \cos(90° - \theta) + i\sin(90° - \theta)$]

11. Express $\frac{-5}{i-\sqrt{3}}$ in the form, $r(\cos\theta + i\sin\theta)$, and hence express $\left(\frac{-5}{i-\sqrt{3}}\right)^6$ in the form $a + ib$.

12. If $z = \cos\theta + i\sin\theta$, show that

(i) $\frac{1}{z} = \cos\theta - i\sin\theta$ (ii) $\frac{1}{z^2} = \cos 2\theta - i\sin 2\theta$

Hence express

(i) $z^2 + \frac{1}{z^2}$ as a multiple of $\cos 2\theta$ (ii) $z^2 - \frac{1}{z^2}$ as a multiple of $\sin 2\theta$.

13. If $z = \cos\theta + i\sin\theta$, show that $\left(z + \frac{1}{z}\right) = 2\cos\theta$.

By expanding $\left(z + \frac{1}{z}\right)^3$, show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

14. Simplify

a) $\frac{(3-3i)^4}{(\sqrt{3}+i)^3}$ b) $\frac{(\sqrt{3}+i)^4}{(1-i)^3}$

15. Calculate

a) $\left\{2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right\}^5$ b) $(1 - i\sqrt{3})^6$

EXERCISES 1-G

Use applications of de Moivre's theorem to prove the following trigonometric identities:

1 $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

2 $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

3 $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

4 $\cos^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

5 $\cos^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

6 **a** Show that $32\cos^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

7 **a** Use de Moivre's theorem to show that $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

b Hence, or otherwise, show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

c Use your answer to part **b** to find, to 2 dp, the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

8. Use de Moivre's theorem to prove the trig. identities

(a) $\sin 2\theta = 2\sin\theta\cos\theta$

(b) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

9. Use De Moivre's Theorem to express cos3θ and sin3θ as polynomials in terms of sinθ and cosθ respectively.

10. Using De Moivre's and the Binomial theorem express cos 4θ as a polynomial in cos θ

11. Find a formula for cos 3θ in terms of powers of cos θ by using De Moivre's theorem. Hence express $\cos^3\theta$ in terms of cos θ and cos 3θ

12. Express sin 3θ as a polynomial in sin θ and hence express $\sin^3\theta$ in terms of sin θ and sin3θ

13. Express $\sin^7\theta$ in terms of the sines multiples of θ.

14. Find formulas for sin 6θ in terms of cos θ and sin θ by using de Movre's theorem

15. Express cos 5θ and sin 5θ as a polynomial in cos θ

16. Use De Moivre's theorem to show that

(a) $\text{Cos}4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $\text{Sin } 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(c) $\text{Cos}5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(d) $\text{Cos}^6\theta = (1/32)[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$

(e) $\text{Cos}^3\theta = (1/4)[\cos 3\theta + 3\cos\theta]$

(f) $\sin^4\theta = (1/4)[\cos 4\theta - 4\cos 2\theta + 3]$

(g) Express $\sin^5\theta$ in terms of sines of multiples of θ.

25. Express $\frac{\sin 5\theta}{\sin\theta}$ as a polynomial in sin θ

26.

a Express $\sin^4\theta$ in the form $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

b Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

27.

Express

a cos 6θ in terms of powers of cos θ,

b $\frac{\sin 6\theta}{\sin\theta}$, $\theta \neq n\pi$, $n \in$ □, in terms of powers of cos θ.

EXERCISES 1-H

1. Express $8i$ in the form $r(\cos\theta + i\sin\theta)$.
Hence find the cube roots of $8i$.

2. Express -1 in the form $r(\cos\theta + i\sin\theta)$.
Hence find the cube roots of -1.

3. Express the following in modulus-argument form and hence use De Moivre's theorem to find the square roots of each:

(i) $4i$ (ii) $2 - 2i\sqrt{3}$.

4. Find the cube roots of $27i$.

5. Solve the equation $z^4 + 1 = 0$.

6. If 1, ω and ω^2 are the cube roots of 1, show that

(i) $(1+\omega^2)^3 = -1$ (ii) $(1-\omega)^2(1+\omega) = 3$.

7. Verify that (i) $(1-\omega+\omega^2)(1+\omega-\omega^2) = 4$ (ii) $\dfrac{\omega}{(1-\omega)^2} = -\dfrac{1}{3}$.

8. Evaluate (i) $(1-\omega)(1-\omega^2)$ (ii) $\omega^2(1+\omega)^2 + \omega^2(1-\omega)^2$.

9. Show that (i) $i(\omega - \omega^2)$ is real

(ii) $\dfrac{\omega^2(\omega-1)^2}{(\omega+1)^3} = 3$

10. Show that $(\omega a + \omega^2 b)(\omega^2 a + \omega b) = a^2 - ab + b^2$

11. Find the four fourth roots of -16.

EXERCISES 1-I

1 Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \square$ and $y \in \square$.

a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

2 Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2\sqrt{3}i = 2$ **f** $z^3 + 16\sqrt{3} + 16i = 0$

3 Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give θ to 2 dp.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

4 a Find the three roots of the equation $(z+1)^3 = -1$.
Give your answers in the form $x + iy$, where $x \in \square$ and $y \in \square$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

5 a Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

b Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

6 a Find the modulus and argument of $-2 - 2\sqrt{3}\mathrm{i}$.

b Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}\mathrm{i} = 0$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

7. Solve the equation $z^3 + 4\sqrt{2} + 4\sqrt{2}\mathrm{i} = 0$.

8. Solve the equation $z^4 = 2 + 2\sqrt{3}\mathrm{i}$.

EXERCISES 1-J (MISCELLANEOUS)

1. (a) Express $\dfrac{(1+2i)^2}{1-i}$ in the form $a + ib$.

(b) Given that $z = -\frac{1}{2} + \frac{1}{2}i$, express $\dfrac{1}{1+z}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta < \pi$.

(c) If 1, ω and ω^2 are the cube roots of unity, show that
$$(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega) = 3.$$

2. (a) Find all possible values of the real numbers a and b which satisfy the equation
$$2 + ai = \frac{6-2i}{b+i}$$

(b) Use De Moivre's theorem to express $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{20}$ in the form $a + ib$.

(c) If $z^2 = -2i$, find the two values for z in the form $a + ib$.

3. (a) Find in the form $a + ib$, the complex number z which satisfies the equation
$$\frac{2z-3}{1-4i} = \frac{3}{1+i}.$$

(b) Given that $z = \sqrt{3} + i$, express in the form $r(\cos\theta + i\sin\theta)$

(i) z^2 (ii) $\frac{1}{z}$

(c) Given that p and q are real and that $1 + 2i$ is a root of the equation $z^2 + (p + 5i)z + q(2 - i) = 0$, find

(i) the values of p and q.
(ii) the other root of the equation.

4. (a) Given that $z_1 = 2 + i$ and $z_2 = -2 + 4i$, find, in the form $a + ib$, the complex number z such that

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

Find also $|z|$.

(b) Express $z = \frac{2}{-1+i}$ in the form $r(\cos\theta + i\sin\theta)$ and hence evaluate

$$\left(\frac{2}{-1+i}\right)^6.$$

(c) If 1, ω and ω^2 are the cube roots of unity, find the value of

$$(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega).$$

5. (a) Find the two real numbers x and y such that

$$x(3 + 4i) - y(1 + 2i) + 5 = 0$$

(b) Without using tables simplify $\dfrac{\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^4}{\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)^5}$.

(c) Find the three roots of the equation $z^3 = -i$.

6. (a) Given that $z = a + ib$, $a,b \in R$, find the possible values of z if $z\bar{z} - 2iz = 7 - 4i$.

(b) If $(a + ib)^2 = 5 - 12i$, find the values of a and b, where $a,b \in R$.

(c) (i) The roots of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the complex numbers p and q.

(ii) If $1 + i$ is a root of the equation $z^2 + (a + 2i)z + 5 + ib = 0$, where a and b are real, find the values of a and b.

7. (a) Find x and $y \in R$ such that $x(1 + i) + 2(1 - 2i)y = 3$.

(b) Show that $1 + i$ is a root of the equation $z^3 - 4z^2 + 6z - 4 = 0$ and find the other roots.

(c) Express $2 - 2i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$ and hence solve the equation

$z^2 = 2 - 2i\sqrt{3}$, giving your answers in the form $a + ib$.

8. (a) Express the complex number $z_1 = \dfrac{11+2i}{3-4i}$ in the form $x + iy$, where x and y are real.
Given that $z_2 = 2 - 5i$, find the real numbers α and β such that $\alpha z_1 + \beta z_2 = -4 + i$.

(b) z is a complex number such that $z = \dfrac{p}{2-i} + \dfrac{q}{1+3i}$ where p and q are real.

If $\arg z = \dfrac{\pi}{2}$ and $|z| = 7$, find the values of p and q.

9. (a) Find the number m such that $|1 - mi| = 2m$, where $m \in R$.

(b) The complex number $z = 2i$. Find the values of a and b such that $(a + ib)^2 = z$.

(c) Express $\dfrac{[\sqrt{3}(\cos\theta + i\sin\theta)]^4}{\cos 2\theta - i\sin 2\theta}$ in the form $r(\cos k\theta + i\sin k\theta)$.

10. (a) The complex numbers $z_1 = \dfrac{a}{1+i}$, $z_2 = \dfrac{b}{1+2i}$, where a and b are real, are such that $z_1 + z_2 = 1$.
Find the values of a and b.

(b) Use De Moivre's theorem to evaluate $(1+i)^n - (1-i)^n$, when $n = 20$.

(c) Show that $(\sin\theta + i\cos\theta)^n = \cos n\left(\dfrac{\pi}{2} - \theta\right) + i\sin n\left(\dfrac{\pi}{2} - \theta\right)$, where $n \in N$.

11. (a) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$ and find the other roots.

(b) If 2, z_1, z_2 and z_3 are the roots of the equation $z^4 = 16$, find the value of
$$(2 - z_1)(2 - z_2)(2 - z_3).$$

(c) Use De Moivre's theorem or the method of induction to show that
$$\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^n = \cos 2n\theta - i\sin 2n\theta.$$

12. (a) The complex number z satisfies the equation $2z\bar{z} - 4z = 3 - 6i$, where $\bar{z}$ is the complex conjugate of z. Find in the form $x + iy$ two possible values of z.

(b) Show that $1 + 2i$ is a root of the equation $z^2 - 3(1+i)z + 5i = 0$ and find the other root.

(c) Expand $\left(z + \dfrac{1}{z}\right)^4$ and $\left(z - \dfrac{1}{z}\right)^4$.

By putting $z = \cos\theta + i\sin\theta$, deduce that
$$\cos^4\theta + \sin^4\theta = \tfrac{1}{4}(\cos 4\theta + 3).$$

13. (a) If $\dfrac{1+2i}{1-i}$ is a root of the equation $ax^2 + bx + c = 0$, $a, b \in R$, find the values of a, b and c.

(b) If 1, ω and ω^2 are the cube roots of 1 and given that $\omega = \omega^2$ and that $1 + \omega + \omega^2 = 0$, find the value of $(2\omega^2 + 5\omega + 2)^6$.

(c) Show that $1 + \cos\theta + i\sin\theta = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$.

14. (a) The complex number z satisfies the equation

$$2z\bar{z} - 4z = 3 - 6i,$$

where $\bar{z}$ is the complex conjugate of z.
Find, in the form $x + iy$, the two possible values of z.

(b) Find the quadratic equation with real coefficients which has

$$2i\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \text{ as one root.}$$

If the roots of this quadratic equation are denoted by α and β, show that $\alpha^6 + \beta^6 = 2^7$.

(c) Solve the equation $\dfrac{z}{3+4i} + \dfrac{z-1}{5i} = \dfrac{5}{3-4i}$.

15. Find the cube roots of 1 + i

16. Find the cube roots of z = 64(cos 30° + i sin 30°)

17. Find the fourth roots of 81i, that is of $81\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

18. Find the roots of $z^2 - (1-i)z + 7i - 4 = 0$ in the form $a + ib$.

19. (a) Compute

$$\int (\cos 2x)^4 \, dx$$

by using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and expanding the fourth power.

(b) Assuming $a \in \mathbb{R}$, compute

$$\int e^{-2x} (\sin ax)^2 \, dx.$$

(same trick: write $\sin ax$ in terms of complex exponentials; make sure your final answer has no complex numbers.)

20. Find all solutions of $z^5 = 6i$.

21. Find the real part of $(\cos 0.7 + i\sin 0.7)^{53}$.

22. Find all complex numbers z, in Cartesian (rectangular) form such that $(z-1)^4 = -1$.

23. Write $(\sqrt{3} + i)^{50}$ in polar and in Cartesian form.

24. Find all fifth roots of -32.

25. Write the following in Cartesian form $a + ib$ where a and b are real and simplified as much as possible:

(a) $\dfrac{1}{1+i} + \dfrac{1}{1-i}$ (b) $e^{2+i\pi/3}$

26. Write all solutions of $z^3 = 8i$ in polar and Cartesian form, simplified as much as possible.
27. Find all complex solutions of the equation $z^5 = 1 + i$.
28. Find the imaginary part of $\frac{2+i}{3-i}$.
29. Find the angle between 0 and 2π that is an argument of $(1-i)^{1999}$.
30. Find all z such that $e^{iz} = 3i$.
31. Write $(1-i)^{100}$ as $a + ib$ with a and b real numbers and simplify your answer.
32. Find the real part of $e^{(5+12i)x}$ where x is real, and simplify your answer.
33. Find all solutions to $z^6 = 8$ and plot them in the complex plane.
34. **Example.** Obtain $\cos 6\theta$ in terms of $\cos\theta$. Hence show that $x = \cos(2k+1)\frac{\pi}{12}$ where $k = 0, 1, 2, 3, 4, 5$ is a solution to the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ and hence deduce that $\cos\frac{\pi}{12}.\cos\frac{5\pi}{12} = \frac{1}{4}$.

EXERCISES 1-K

1. Express e^{1-i} in rectangular form accurately to three decimal places.
2. Compute sin(1-i)
3. Evaluate cos(1+2i)
4. Evaluate cos(π-i)
5. Find all values of ln2
6. Find all values of ln(-1)
7. Find the principal value of ln(1+i)
8. Find the principal value of $(1+i)^{1-i}$
9. Find the principal value of $(1+i)^{2-i}$
10. Find the principal value of 2^i
11. Find all values of $(i)^{1/2}$
12. Find all values of $(i)^{2/3}$
13. Find and simplify cot (π -i ln3)

EXERCISES 1-L

Evaluate the following integral by using complex exponentials

a. $\int \sin 3x \sin 5x \, dx$
b. $\int \cos 2x \cos 4x \, dx$
c. $\int \sin 2x \cos 4x \, dx$
d. $\int \sin^4 x \, dx$
e. $\int \sin^6 x \, dx$
f. $\int \sin^2 x \cos^2 x \, dx$
g. $\int \cos^2 x - \sin^2 x \, dx$
h. $\int \cos^4 x - \sin^4 x \, dx$
i. $\int \cos 2x \sin 2x \, dx$
j. $\int_0^{\pi/2} \cos^4 x \, dx$
k. $\int_0^{\pi} \cos 3x \cos 5x \, dx$
l. $\int_0^{\pi/4} \cos^2 x \sin^2 x \, dx$
m. $\int \cos^8 x \, dx$
n. $\int \cos^4 x \sin^2 x \, dx$
o. $\int \frac{\tan^4 x}{\sec^4 x} \, dx$

P. $\int e^x \cos 2x \, dx$
q. $\int \cos^4 x \sin^4 x \, dx.$
r. $\int \sin^2(2x) \cos(3x) \, dx$

s. $\int e^{3x} \sin 2x \, dx$

EXERCISES 1-M (MISCELLANEOUS)

1 If $z = x + jy$, where x and y are real, find the values of x and y when $\frac{3z}{1-j} + \frac{3z}{j} = \frac{4}{3-j}$.

2 In the Argand diagram, the origin is the centre of an equilateral triangle and one vertex of the triangle is the point $3 + j\sqrt{3}$. Find the complex numbers representing the other vertices.

3 Express $2 + j3$ and $1 - j2$ in polar form and apply DeMoivre's theorem to evaluate $\frac{(2+j3)^4}{1-j2}$. Express the result in the form $a + jb$ and in exponential form.

4 Find the fifth roots of $-3 + j3$ in polar form and in exponential form.

5 Express $5 + j12$ in polar form and hence evaluate the principal value of $\sqrt[3]{(5 + j12)}$, giving the results in the form $a + jb$ and in the form $re^{j\theta}$.

6 Determine the fourth roots of -16, giving the results in the form $a + jb$.

7 Find the fifth roots of -1, giving the results in polar form. Express the principal root in the form $re^{j\theta}$.

8 Determine the roots of the equation $x^3 + 64 = 0$ in the form $a + jb$, where a and b are real.

9 Determine the three cube roots of $\frac{2-j}{2+j}$ giving the result in modulus/argument form. Express the principal root in the form $a + jb$.

10 Show that the equation $z^3 = 1$ has one real root and two other roots which are not real, and that, if one of the non-real roots is denoted by ω, the other is then ω^2. Mark on the Argand diagram the points which represent the three roots and show that they are the vertices of an equilateral triangle.

11 Determine the fifth roots of $(2 - j5)$, giving the results in modulus/argument form. Express the principal root in the form $a + jb$ and in the form $re^{j\theta}$.

12 Solve the equation $z^2 + 2(1 + j)z + 2 = 0$, giving each result in the form $a + jb$, with a and b correct to 2 places of decimals.

13 Express $e^{1-j\pi/2}$ in the form $a + jb$.

14 Obtain the expansion of $\sin 7\theta$ in powers of $\sin\theta$.

15 Express $\sin^6 x$ as a series of terms which are cosines of angles that are multiples of x.

16 If $z = x + jy$, where x and y are real, show that the locus $\left|\dfrac{z-2}{z+2}\right| = 2$ is a circle and determine its centre and radius.

17 If $z = x + jy$, show that the locus $\arg\left\{\dfrac{z-1}{z-j}\right\} = \dfrac{\pi}{6}$ is a circle. Find its centre and radius.

18 If $z = x + jy$, determine the Cartesian equation of the locus of the point z which moves in the Argand diagram so that

$|z + j2|^2 + |z - j2|^2 = 40.$

19 If $z = x + jy$, determine the equations of the two loci:

(a) $\left|\dfrac{z+2}{z}\right| = 3$ and (b) $\arg\left\{\dfrac{z+2}{z}\right\} = \dfrac{\pi}{4}$

20 If $z = x + jy$, determine the equations of the loci in the Argand diagram, defined by:

(a) $\left|\dfrac{z+2}{z-1}\right| = 2$ and (b) $\arg\left\{\dfrac{z-1}{z+2}\right\} = \dfrac{\pi}{2}$

21 Prove that:

(a) if $|z_1 + z_2| = |z_1 - z_2|$, the difference of the arguments of z_1 and z_2 is $\dfrac{\pi}{2}$

(b) if $\arg\left\{\dfrac{z_1 + z_2}{z_1 - z_2}\right\} = \dfrac{\pi}{2}$, then $|z_1| = |z_2|$

22 If $z = x + jy$, determine the loci in the Argand diagram, defined by:

(a) $|z + j2|^2 - |z - j2|^2 = 24$

(b) $|z + jk|^2 + |z - jk|^2 = 10k^2 \quad (k > 0)$

23 If $z = x + jy$, find the locus $\arg(z + 1) = \dfrac{\pi}{3}$.

24 If $z = x + jy$, find the equation of the locus $\arg(z^2) = -\dfrac{\pi}{4}$.

25 If $z = x + jy$, find the equation of the locus $\left|\dfrac{z+1}{z-1}\right| = 2$.

26. If $z_1 = 1 + i$ & $z_2 = 2 + 3i$ find the locus of z if $|z - z_1| = |z - z_2|$.

27. Sketch the locus of $\Re(z + iz) < 2$.

28. Describe in geometric terms, the curve described by $2|z| = z + \overline{z} + 4$.

29. Sketch the curve: (i) $\Re(z^2) = 3$ (ii) $\Im(z^2) = 4$.

Show algebraically that $|z - 2 - i| = 4$ represents a circle with radius 4 units and centre $(2, 1)$.

30. If $\arg(z + 3 + 2i) = \frac{3\pi}{4}$, sketch the locus of z on an Argand diagram.

Find the Cartesian equation of this locus.

31. Find the loci in the complex plane given by (a) $\text{Re}(z) = 2$, (b) $\left|\frac{z+1}{z-1}\right| = 2$.