11

LOGARITHMS

11.1 INTRODUCTION

Logarithms are used to make the long and complicated calculations easy.

Consider $3^4 = 81$, this is the exponential form of representing relation between three numbers 3, 4 and 81. Now the same relation between 3, 4 and 81 can be written as

$\log_3^{81} = 4$ (read as : logarithm of 81 at base 3 is 4).

Thus : $3^4 = 81 \Leftrightarrow \log_3^{81} = 4$

Definition : If a, b and c are three real numbers such that $a \neq 1$ and $a^b = c$ then b is called logarithm of c at the base a and is written as $\log_a^c = b$; read as log of c at the base a is b.

$$a^b = c \Leftrightarrow \log_a^c = b$$

11.2 INTERCHANGING

(Logarithmic form vis-à-vis exponential form)

$a^b = c$ is called the ***exponential form***

and, $\log_a c = b$ is called the ***logarithmic form.***

i.e., (i) $2^{-3} = 0{\cdot}125$ [Exponential form]

$\Rightarrow$ log of $0{\cdot}125$ to the base $2 = -3$

i.e., $\log_2 0{\cdot}125 = -3$ [Logarithmic form]

(ii) $\log_{64} 8 = \frac{1}{2}$ [Logarithmic form]

$\Rightarrow$ log of 8 to the base $64 = \frac{1}{2}$

i.e. $(64)^{\frac{1}{2}} = 8$ [Exponential form] and so on.

Similarly :

If x is positive;

(iii) $x^0 = 1 \Rightarrow$ log of 1 to the base $x = 0$ *i.e.* $\log_x 1 = 0$

(iv) $x^1 = x \Rightarrow$ log x to the base $x = 1$ *i.e.* $\log_x x = 1$ and so on.

From (iii), we get : *the logarithm of 1 to any base is zero.*

i.e. $\log_5 1 = 0$; $\log_{10}^1 = 0$; $\log_a 1 = 0$ and so on.

From (iv), we get : *the logarithm of any number to the same base is always one.*

i.e. $\log_5 5 = 1$; $\log_{10} 10 = 1$; $\log_a a = 1$ and so on.

1 Find : (i) the logarithm of 1000 to the base 10.

(ii) the logarithm of $\frac{1}{9}$ to the base 3.

Solution :

(i) Let $\log_{10}1000 = x \Rightarrow 10^x = 1000$

$\Rightarrow 10^x = 10^3 \Rightarrow x = 3$

$\therefore \log_{10}1000 = 3$ **Ans.**

(ii) Let $\log_3 \frac{1}{9} = x \Rightarrow 3^x = \frac{1}{9}$

$\Rightarrow 3^x = 3^{-2} \Rightarrow x = -2$

$\therefore \log_3 \frac{1}{9} = -2$ **Ans.**

2 Find x, if : (i) $\log_2 x = -2$ (ii) $\log_4(x + 3) = 2$ (iii) $\log_9 27 = x$

(iv) $\log_x 64 = \frac{3}{2}$ (v) $\log_2 (x^2 - 4) = 5$

Solution :

(i) $\log_2 x = -2 \Rightarrow 2^{-2} = x$

$\Rightarrow x = \frac{1}{4}$ **Ans.**

(ii) $\log_4 (x + 3) = 2 \Rightarrow 4^2 = x + 3$

$\Rightarrow x = 16 - 3 = 13$ **Ans.**

(iii) $\log_9 27 = x \Rightarrow 9^x = 27$

$\Rightarrow 3^{2x} = 3^3 \Rightarrow x = \frac{3}{2}$ **Ans.**

(iv) $\log_x 64 = \frac{3}{2} \Rightarrow x^{\frac{3}{2}} = 64$

$\Rightarrow x = (64)^{\frac{2}{3}} = (2^6)^{\frac{2}{3}} = 16$ **Ans.**

(v) $\log_2 (x^2 - 4) = 5 \Rightarrow 2^5 = x^2 - 4$

$\Rightarrow x^2 = 32 + 4 = 36$

$\Rightarrow x = \pm 6$ **Ans.**

EXERCISE 11(A)

1. Express each of the following in *logarithmic form :*

   (i) $5^3 = 125$ (ii) $3^{-2} = \frac{1}{9}$

   (iii) $10^{-3} = 0{\cdot}001$ (iv) $(81)^{\frac{3}{4}} = 27$

2. Express each of the following in *exponential form :*

   (i) $\log_8 0{\cdot}125 = -1$ (ii) $\log_{10} 0{\cdot}01 = -2$

   (iii) $\log_a A = x$ (iv) $\log_{10} 1 = 0$

3. Solve for x : $\log_{10} x = -2$.

4. Find the logarithm of :

   (i) 100 to the base 10

   (ii) 0·1 to the base 10

   (iii) 0·001 to the base 10

   (iv) 32 to the base 4

   (v) 0·125 to the base 2

   (vi) $\frac{1}{16}$ to the base 4

(vii) 27 to the base 9

(viii) $\frac{1}{81}$ to the base 27

5. State, true or false :

(i) If $\log_{10} x = a$, then $10^x = a$.

(ii) If $x^y = z$, then $y = \log_z x$.

(iii) $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$.

6. Find x, if :

(i) $\log_3 x = 0$ (ii) $\log_x 2 = -1$

(iii) $\log_9 243 = x$ (iv) $\log_5 (x-7) = 1$

(v) $\log_4 32 = x - 4$ (vi) $\log_7 (2x^2 - 1) = 2$

7. Evaluate :

(i) $\log_{10} 0{\cdot}01$ (ii) $\log_2 \frac{1}{8}$

(iii) $\log_5 1$ (iv) $\log_5 125$

(v) $\log_{16} 8$ (vi) $\log_{0\cdot5} 16$

11.3 LAWS OF LOGARITHM WITH USE

First Law (Product Law) :

The *logarithm of a product at any non-zero base* is equal to the *sum of the logarithms* of its factors at the same base.

i.e. $$\log_a (m \times n) = \log_a m + \log_a n$$

$$\log_x (m \times n \times p) = \log_x m + \log_x n + \log_x p \quad \text{and so on.}$$

Remember : $\log_a (m + n) \neq \log_a m + \log_a n$

Second Law (Quotient Law) :

The *logarithm of a fraction* at any non-zero base is equal to the *difference between the logarithm of the numerator minus the logarithm of the denominator, both at the same base.*

i.e. $$\log_a \frac{m}{n} = \log_a m - \log_a n$$

Remember : $$\frac{\log_a m}{\log_a n} \neq \log_a m - \log_a n$$

Third Law (Power Law) :

The *logarithm of a power of a number at any non-zero base* is equal to the *logarithm of the number (at the same base) multiplied by the power.*

i.e. $$\log_a (m)^n = n \log_a m$$

Corollary : Since $\sqrt[n]{m} = m^{\frac{1}{n}}$

$$\therefore \quad \log_a \sqrt[n]{m} = \log_a m^{\frac{1}{n}} = \frac{1}{n} \log_a m$$

1. Logarithms to the **base 10** are known as ***common logarithms.***
2. If **no base is given,** the **base is** always taken as **10,**
   i.e. $\log 8 = \log_{10} 8$; $\log a = \log_{10} a$; $\log 10 = \log_{10} 10$ and so on.
3. $\log_{10} 1 = 0$; $\log_{10} 10 = 1$;
   $\log_{10} 100 = 2$ [$\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2 \times 1 = 2$]
   Similarly, $\log_{10} 1000 = 3$; $\log_{10} 10000 = 4$ and so on.

11.4 EXPANSION OF EXPRESSION WITH THE HELP OF LAWS OF LOGARITHM

Let $y = \frac{a^4 \times b^2}{c^3} \Rightarrow \log y = \log \frac{a^4 \times b^2}{c^3}$

i.e. $\log y = \log(a^4 \times b^2) - \log c^3$ $\quad [\because \log \frac{m}{n} = \log m - \log n]$

$= \log a^4 + \log b^2 - \log c^3$ $\quad [\because \log m \times n = \log m + \log n]$

$= 4 \log a + 2 \log b - 3 \log c$ $\quad [\because \log m^n = n \log m]$

$\therefore$ $\log y = 4 \log a + 2 \log b - 3 \log c$ is the logarithmic expansion of the given expression $y = \frac{a^4 \times b^2}{c^3}$

Similarly,

$$m = \frac{3^x}{5^y \times 8^z} \Rightarrow \log m = \log 3^x - \log (5^y \times 8^z)$$
$$= x \log 3 - [\log 5^y + \log 8^z]$$
$$= x \log 3 - y \log 5 - z \log 8$$
$$\Rightarrow \log m = x \log 3 - y \log 5 - z \log 8$$

Conversely :

(i) $\log V = \log \pi + 2 \log r + \log h - \log 3$

$$\Rightarrow \log V = \log \pi + \log r^2 + \log h - \log 3$$
$$= \log \frac{\pi r^2 h}{3}$$
$$\Rightarrow V = \frac{\pi r^2 h}{3}$$

(ii) $\log F = \log G + \log m_1 + \log m_2 - 2 \log r$

$$\Rightarrow \log F = \log G + \log m_1 + \log m_2 - \log r^2$$
$$= \log \frac{G m_1 m_2}{r^2}$$
$$\Rightarrow F = \frac{G m_1 m_2}{r^2}$$ and so on.

3 Express $\log_{10} \sqrt[5]{108}$ in terms of $\log_{10} 2$ and $\log_{10} 3$.

Solution :

$\log_{10} \sqrt[5]{108} = \log_{10}(108)^{\frac{1}{5}}$ $\quad [\sqrt[n]{m} = m^{\frac{1}{n}}]$

$= \frac{1}{5} \log_{10} 108$ $\quad [\log_{10} n^m = m \log_{10} n]$

$= \frac{1}{5} \log_{10}(2^2 \times 3^3)$ $\quad [108 = 2 \times 2 \times 3 \times 3 \times 3 = 2^2 \times 3^3]$

$= \frac{1}{5} [\log_{10} 2^2 + \log_{10} 3^3]$ $\quad [\log_{10} m \times n = \log_{10} m + \log_{10} n]$

$= \frac{1}{5} [2\log_{10} 2 + 3\log_{10} 3]$ **Ans.**

4 Express as a single logarithm : $2 + \frac{1}{2}\log_{10}9 - 2\log_{10}5$

Solution :

$= \log_{10}100 + \frac{1}{2}\log_{10}3^2 - \log_{10}5^2$ [$\log_{10}100 = 2$; $\log 9 = \log 3^2$ and $2\log 5 = \log 5^2$]

$= \log_{10}100 + \frac{1}{2} \times 2\log_{10}3 - \log_{10}25$

$= \log_{10}\frac{100\times3}{25} = \log_{10}12$ [$\log a + \log b - \log c = \log\frac{a\times b}{c}$] **Ans.**

5 Without using the log tables, evaluate : $2\log_{10}5 + \log_{10}8 - \frac{1}{2}\log_{10}4$

Solution :

$= \log_{10}5^2 + \log_{10}8 - \log_{10}(4)^{\frac{1}{2}}$

$= \log 25 + \log 8 - \log 2$ [If no base is written, it means the base is 10]

$= \log\frac{25\times8}{2} = \log 100 = 2$ [$\log_{10}100 = 2$] **Ans.**

6 Find x, if : (i) $\log_{10}(x + 5) = 1$.

(ii) $\log_{10}(x + 1) + \log_{10}(x - 1) = \log_{10}11 + 2\log_{10}3$

Solution :

(i) $\Rightarrow \log_{10}(x + 5) = \log_{10}10$

$\Rightarrow x + 5 = 10 \Rightarrow x = 5$ **Ans.**

(ii) $\Rightarrow \log_{10}(x + 1)(x - 1) = \log_{10}11 + \log_{10}3^2$

$\Rightarrow \log(x^2 - 1) = \log(11 \times 9)$

$\Rightarrow x^2 - 1 = 99$

$\therefore x^2 = 100$ and, $x = 10$ **Ans.**

7 If $\log 2 = 0{\cdot}3010$ and $\log 3 = 0{\cdot}4771$, find the value of :

(i) $\log 6$ (ii) $\log 5$ (iii) $\log\sqrt{24}$

Solution :

(i) $\log 6 = \log 2 \times 3 = \log 2 + \log 3 = 0{\cdot}3010 + 0{\cdot}4771 = 0{\cdot}7781$ **Ans.**

(ii) $\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - 0{\cdot}3010 = 0{\cdot}6990$ **Ans.** [$\because \log 10 = 1$]

(iii) $\log\sqrt{24} = \log(24)^{\frac{1}{2}} = \frac{1}{2}\log(2^3 \times 3)$ [$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$]

$= \frac{1}{2}[3 \times \log 2 + \log 3]$

$= \frac{1}{2}[3 \times 0{\cdot}3010 + 0{\cdot}4771] = 0{\cdot}69005$ **Ans.**

EXERCISE 11(B)

1. Express in terms of log 2 and log 3:
   (i) log 36 (ii) log 144
   (ii) log 4·5 (iv) $\log \frac{26}{51} - \log \frac{91}{119}$
   (v) $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243}$
2. Express each of the following in a form free from logarithm :
   (i) $2 \log x - \log y = 1$
   (ii) $2 \log x + 3 \log y = \log a$
   (iii) $a \log x - b \log y = 2 \log 3$
   (iv) $\log V = 2 \log 2 - \log 3 + \log \pi + 3 \log r$
   (v) $\log F = \log G + \log m_1 + \log m_2 - 2 \log d$
3. Express as a single logarithm:
   (i) log 4 + 2 log 3
   (ii) 2 log 9 – log 18
   (iii) 3 log 2 – 2 log 3
   (iv) 1 – 2 log 5 + 3 log 2
4. Evaluate each of the following without using tables :
   (i) log 5 + log 8 – 2 log 2
   (ii) $\log_{10} 8 + \log_{10} 25 + 2 \log_{10} 3 - \log_{10} 18$
   (iii) $\log 4 + \frac{1}{3} \log 125 - \frac{1}{5} \log 32$
5. Prove that :
   $$2 \log \frac{15}{18} - \log \frac{25}{162} + \log \frac{4}{9} = \log 2.$$
6. Find x, if :
   $$x - \log 48 + 3 \log 2 = \frac{1}{3} \log 125 - \log 3.$$
7. Express $\log_{10} 2 + 1$ in the form of $\log_{10} x$.
8. Solve for x :
   (i) $\log_{10} (x - 10) = 1$
   (ii) $\log (x^2 - 21) = 2$
   (iii) $\log (x - 2) + \log (x + 2) = \log 5$
   (iv) $\log (x + 5) + \log (x - 5) = 4 \log 2 + 2 \log 3$
   (v) $\log (x + 4) - \log (x - 4) = \log 2$
   (vi) $\log (x + 3) - \log (x - 3) = 1$
9. Solve for x :
   (i) $\frac{\log 81}{\log 27} = x$ (ii) $\frac{\log 128}{\log 32} = x$
   (iii) $\frac{\log 64}{\log 8} = \log x$ (iv) $\frac{\log 225}{\log 15} = \log x$

   (i) $\frac{\log 81}{\log 27} = x$

   $\Rightarrow x = \frac{\log 3^4}{\log 3^3} = \frac{4 \log 3}{3 \log 3} = \frac{4}{3}$ **Ans.**

10. Given $\log x = m + n$ and $\log y = m - n$, express the value of $\log \frac{10x}{y^2}$ in terms of m and n.
11. State, true or false :
    (i) log 1 × log 1000 = 0
    (ii) $\frac{\log x}{\log y} = \log x - \log y$
    (iii) If $\frac{\log 25}{\log 5} = \log x$, then $x = 2$
    (iv) $\log x \times \log y = \log x + \log y$
12. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$; express each of the following in terms of 'a' and 'b' :
    (i) log 12 (ii) log 2·25 (iii) $\log 2\frac{1}{4}$
    (iv) log 5·4 (v) log 60 (vi) $\log 3\frac{1}{8}$
13. If log 2 = 0·3010 and log 3 = 0·4771; find the value of :
    (i) log 12 (ii) log 1·2
    (iii) log 3·6 (iv) log 15
    (v) log 25 (vi) $\frac{2}{3} \log 8$
14. Given $2 \log_{10} x + 1 = \log_{10} 250$, find :
    (i) x (ii) $\log_{10} 2x$

8 If $\log_{10}4 = 0{\cdot}6020$; find the value of :

(i) $\log_{10}8$ (ii) $\log_{10}2{\cdot}5$

Solution :

$$\text{If } \log_{10}4 = 0{\cdot}6020 \Rightarrow 2\log 2 = 0{\cdot}6020 \qquad [\because \log 4 = \log 2^2 = 2\log 2]$$

$$\Rightarrow \log 2 = \frac{0{\cdot}6020}{2} = 0{\cdot}3010$$

$\therefore$ (i)

$$\log_{10}8 = \log_{10}2^3$$
$$= 3\log 2 = 3 \times 0{\cdot}3010 = \mathbf{0{\cdot}9030} \quad \textbf{Ans.}$$

(ii)

$$\log_{10}2{\cdot}5 = \log\frac{25}{10} = \log\frac{5}{2} = \log\frac{10}{2\times2}$$
$$= \log 10 - 2\log 2$$
$$= 1 - 2 \times 0{\cdot}3010 = \mathbf{0{\cdot}3980} \quad \textbf{Ans.}$$

9 Given $\log_{10}x = a$ and $\log_{10}y = b$.

(i) Write down 10^{a-1} in terms of x.

(ii) Write down 10^{2b} in terms of y.

(iii) If $\log_{10}P = 2a - b$; express P in terms of x and y.

Solution :

(i) $\log_{10}x = a \Rightarrow 10^a = x$

$$\therefore \quad 10^{a-1} = \frac{10^a}{10^1} = \frac{x}{10} \quad \textbf{Ans.}$$

(ii) $\log_{10}{}^{y} = b \Rightarrow 10^b = y$

$$\therefore \quad 10^{2b} = (10^b)^2 = y^2 \quad \textbf{Ans.}$$

(iii) $\log_{10}P = 2a - b$

$$\Rightarrow \log_{10}P = 2\log_{10}x - \log_{10}y$$
$$\Rightarrow \log P = \log x^2 - \log y$$
$$\Rightarrow \log P = \log\frac{x^2}{y}$$
$$\therefore \quad P = \frac{x^2}{y} \quad \textbf{Ans.}$$

EXERCISE 11(C)

1. If $\log_{10}8 = 0{\cdot}90$; find the value of :
   (i) $\log_{10}4$ (ii) $\log\sqrt{32}$
   (iii) $\log 0{\cdot}125$
2. If $\log 27 = 1{\cdot}431$, find the value of :
   (i) $\log 9$ (ii) $\log 300$
3. If $\log_{10}a = b$, find 10^{3b-2} in terms of a.
4. If $\log_5 x = y$, find 5^{2y+3} in terms of x.
5. Given: $\log_3 m = x$ and $\log_3 n = y$.
   (i) Express 3^{2x-3} in terms of m.

(ii) Write down $3^{1-2y+3x}$ in terms of m and n.

(iii) If $2 \log_3 A = 5x - 3y$; find A in terms of m and n.

6. Simplify :

(i) $\log (a)^3 - \log a$ (ii) $\log (a)^3 \div \log a$

7. If $\log (a + b) = \log a + \log b$, find a in terms of b.

8. If $l = \log \frac{a^2}{bc}$, $m = \log \frac{b^2}{ca}$ and $n = \log \frac{c^2}{ab}$, find the value of $l + m + n$.

9. Prove that :

(i) $(\log a)^2 - (\log b)^2 = \log \left(\frac{a}{b}\right) . \log (ab)$

(ii) If $a \log b + b \log a - 1 = 0$, then $b^a . a^b = 10$

10. (i) If $\log (a + 1) = \log (4a - 3) - \log 3$; find a.

(ii) If $2 \log y - \log x - 3 = 0$, express x in terms of y.

(iii) Prove that : $\log_{10} 125 = 3(1 - \log_{10} 2)$.

EXERCISE 11(D)

1. If $\frac{3}{2} \log a + \frac{2}{3} \log b - 1 = 0$, find the value of $a^9.b^4$.

2. If $x = 1 + \log 2 - \log 5$, $y = 2 \log 3$ and $z = \log a - \log 5$; find the value of a if $x + y = 2z$.

3. If $x = \log 0{\cdot}6$; $y = \log 1{\cdot}25$ and $z = \log 3 - 2 \log 2$, find the values of :

(i) $x + y - z$ (ii) 5^{x+y-z}

4. If $a^2 = \log x$, $b^3 = \log y$ and $3a^2 - 2b^3 = 6 \log z$, express y in terms of x and z.

5. If $\log \frac{a-b}{2} = \frac{1}{2} (\log a + \log b)$, show that: $a^2 + b^2 = 6ab$.

6. If $a^2 + b^2 = 23ab$, show that :
$\log \frac{a+b}{5} = \frac{1}{2} (\log a + \log b)$.

7. If $m = \log 20$ and $n = \log 25$, find the value of x, so that : $2 \log (x - 4) = 2m - n$.

8. Solve for x and y; if $x > 0$ and $y > 0$:
$\log xy = \log \frac{x}{y} + 2 \log 2 = 2$.

9. Find x, if :

(i) $\log_x 625 = -4$ (ii) $\log_x (5x - 6) = 2$.

10. If $x = 1 + \log 2 - \log 5$, $y = 2 \log 3$, $z = \log 3m - \log 5$ and $x + y = 2z$; find the value of m.

11. If $\log_2(x + y) = \log_3(x - y) = \frac{\log 25}{\log 0{\cdot}2}$, find the values of x and y.

12. Given : $\frac{\log x}{\log y} = \frac{3}{2}$ and $\log (xy) = 5$; find the values of x and y.

13. Given $\log_{10} x = 2a$ and $\log_{10} y = \frac{b}{2}$.

(i) Write 10^a in terms of x.

(ii) Write 10^{2b+1} in terms of y.

(iii) If $\log_{10}^{P} = 3a - 2b$, express P in terms of x and y.

14. Solve :
$\log_5(x + 1) - 1 = 1 + \log_5(x - 1)$.

15. Solve for x, if :
$\log_x 49 - \log_x 7 + \log_x \frac{1}{343} + 2 = 0$.